Asymptotic expansions of the complete elliptic integrals about unitary modulus

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Abstract

Power series in the complementary modulus for the first and second complete elliptic integrals are deduced in terms of binomial series, by exploiting a suitable decomposition of the integration domain. This approach appears to be straightforward, with respect to the standard one. However, despite the procedure is simple, it needs some non-trivial results about binomial series proved in the appendix. Numerical performances of the expansions are also discussed and compared with existing alternative expansions.

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1. Introduction.

The present paper is aimed to the numerical evaluation of the first ($K$) and second ($E$) complete elliptic integrals:

\begin{align*}
K(k) &= \int_0^1 dt \frac{1}{(1-t^2)^{1/2} (1-k^2 t^2)^{1/2}}, \\
E(k) &= \int_0^1 dt \frac{(1-k^2 t^2)^{1/2}}{(1-t^2)^{1/2}}
\end{align*}

in a left neighbourhood of the point $k = 1$. Basic properties of these functions are discussed in \cite{1} and in \cite{2}, among many others.

The computation of these integrals is often needed in the physical applications: numerical techniques and also fortran routines for evaluating
K and E (with k not too close to 1) can be found in Section 18.3 of [3].
The sample case which motivates the present study is the computation of the surface integrals in \(dS(y)\) of the three dimensional Green function
\[ G(x - y) = -1/(4\pi |x - y|) \]
and of its normal derivative, carried out for an axisymmetric surface. Named as \((\xi, \sigma)\) and \((\eta, \omega)\) the axial and radial coordinates of \(x\) and \(y\), respectively, the above double integrals reduce to axial \((\eta)\) one and contain the complete elliptic integrals \(K(k)\) and \(E(k)\), with modulus \(k = \{4\sigma\omega/[(\xi - \eta)^2 + (\sigma + \omega)^2]\}^{1/2}\).

As far as the point \(x\) is sufficiently far from \(y\), the modulus is reasonably smaller than the unity, so that accurate and efficient numerical evaluations of these integrals are obtained by their classical series representations \(^a\) (see for example the formulae \(I_a\) and \(II_a\) in [4]):

\[
(2) \quad K(k) = \frac{\pi}{2} \sum_{m=0}^{\infty} b_m^{-2} k^{2m}, \quad E(k) = -\frac{\pi}{4} \sum_{m=0}^{\infty} \frac{b_m^{-2}}{m - 1/2} k^{2m}.
\]

Hereafter, they will be approximated by the corresponding partial sums of order \(n\). This number will be chosen by enforcing that the \((n + 1)\)th term (in absolute value) is the first smaller than the threshold value \(10^{-16}\). In Fig. 1 \(n\) is drawn vs. \(k\) on a left neighbourhood of \(k = 1\) for the partial sums of the series of \(K\) (red line) and of \(E\) (blue).

On the contrary, in correspondence to points \(x\) near \(y\) (\(\xi \rightarrow \eta\) and \(\sigma \rightarrow \omega\)) the modulus \(k\) approaches unity and the convergence of the series (2) becomes worse. Indeed, consider large values of the index \(m\) in the series (2). The binomial coefficient \(b_m^{-}\) can be approximated with \((-1)^m/\sqrt{\pi m}\), so that the \(m\)th terms of the series of \(K\) and \(E\) approach \(k^{2m}/(2m)\) and \(-k^{2m}/(4m^2)\), respectively. As a consequence, as \(k \rightarrow 1\) the order \(n\) of the partial sum of \(K\) diverges, while the one of \(E\) remains finite, even if quite large. As a sample, at \(k = 0.9999\) one finds \(n = 120000\) for the series of \(K\) and \(n = 67309\) for the one of \(E\).

In order to recover computational efficiency (and also accuracy, see Section 4), different approximations are needed in a left neighbourhood of the point \(k = 1\). In particular, our attention is here focused on the expansions of the complete elliptic integrals \(K\) and \(E\) in power series of the complementary modulus \(k' = (1 - k^2)^{1/2}\). As it is well known, this problem has been solved in [4] by means of a sophisticated mathematical technique

\(^a\) In the formulae (2) and hereafter the symbols:
\[
\left( \frac{\mp 1/2}{m} \right) := b_m^\pm, \quad \left( \frac{3/2}{m} \right) := c_m
\]
are used for short.
which uses the ordinary differential equations (ODE) satisfied by $K$ and $E$ as functions of $k'$ (equations (6") and (7") in [4], rewritten here for the Reader convenience):

$$
\begin{align*}
&k'(1 - k'^2) \frac{d^2 K}{dk'^2} + (1 - 3k'^2) \frac{dK}{dk'} - k'K = 0 \\
&k'(1 - k'^2) \frac{d^2 E}{dk'^2} - (1 + k'^2) \frac{dE}{dk'} + k'E = 0
\end{align*}
$$

as well as the qualitative theory of the systems of ODE. In particular, this theory is employed in writing the power series solution of the above system in the following form:

$$
(3) \quad K(k') = P_K(k') \log k' + Q_K(k') , \quad E(k') = P_E(k') \log k' + Q_E(k') ,
$$

$P_K, Q_K, P_E$ and $Q_E$ being power series in $k'^2$. In the solution (3), the coefficient of $k'^0$ in $P_K$ is $-1$, while it vanishes in $P_E$, according to the different behaviour of the two integrals (1) as $k' \to 0$ ($K$ diverges as $- \log k'$, while $E$ is finite for $k' \to 0$). The other coefficients in the first two terms of $P_K, Q_K, P_E$ and $Q_E$ are directly calculated, while the successive ones are evaluated by means of recurrence formulae. The asymptotic expansions $I_b$ and $II_b$ in [4] follow.

In the present paper a simpler way to evaluate the complete elliptic integrals (1) in a left neighbourhood of $k = 1$ is described. It is based still on the binomial series and exploits decompositions of the integration domain, according to the behaviour of the integrand functions. It leads to representations closely resembling the ones in equation (3), without resorting
to differential equations or hypergeometric functions. Moreover, it leads to rather efficient computations of the above integrals, here performed by using FORTRAN programming language. Numerical tests show that the present expansions behave in terms of accuracy and computational cost as the ones in [4], so that they have significance essentially from a theoretical point of view. The representation of \( K \) is discussed in Section 2, while the one of \( E \) in Section 3. They need some formulae involving binomial coefficients, shortly proved in A. Finally, a brief discussion of the numerical performances of these representations together with some concluding remarks are offered in Section 4.

2. Complete elliptic integral of first kind.

In order to write an asymptotic expansion for \( K \) in terms of the complementary modulus \( k' := (1 - k^2)^{1/2} \), the value \( t^* = 1/(1 + k'^2)^{1/2} \) of \( t \) such that \( 1 - t^2 = k'^2t^2 \) is introduced. The integration range in the first integral in equation (1) is split in the union of the intervals \((0, t^*)\) (where \( 1 - t^2 > k'^2t^2 \)) and \((t^*, 1)\) (where \( 1 - t^2 < k'^2t^2 \)):

\[
K(k) = \int_0^{t^*} \frac{dt}{(1 - t^2)^{1/2}} \left( \frac{1}{1 - t^2 + k'^2t^2} \right)^{1/2} + \int_{t^*}^1 \frac{dt}{(1 - t^2)^{1/2}} \left( \frac{1}{1 - k'^2t^2} \right)^{1/2}.
\]

The two integrals \( K' \) and \( K'' \) are handled in different ways, as briefly discussed below.

The integrand \( K' \) is expanded in a binomial series of argument \( k'^2t^2/(1 - t^2) \):

\[
K'(k) = \sum_{l=0}^{\infty} b_l k'^{2l} \left( \int_0^{t^*} \frac{t^{2l}}{(1 - t^2)^{l+1}} \right)_{I_l} = \omega^* \sum_{j=0}^{\infty} s_j k'^{2j} + I_0(k') F(k')
\]

with \( \omega^* = 1/t^* \) and the integrals are evaluated by means of the recurrence relations \( I_0 = -\log k' + \log(1 + \omega^*) \), \( I_l = [\omega^* - (2l - 1)k'^2I_{l-1}] / (2l) \) for \( l \geq 1 \). They give the integrals:

\[
I_1 = \frac{\omega^*}{2} - \frac{1}{2} k'^2 I_0
\]

\[
I_2 = \frac{\omega^*}{4} - \frac{3}{8} \omega^* k'^2 + \frac{3}{8} k'^4 I_0
\]

\[
I_3 = \frac{\omega^*}{6} - \frac{5}{24} \omega^* k'^2 + \frac{5}{16} \omega^* k'^4 - \frac{5}{18} k'^6 I_0
\]
and so on. An inspection of the above sequence leads to the $l$th integral:

$$I_l = \frac{1}{2l} a s - \frac{2l-1}{2l(2l-2)} a^{s-2} + \frac{(2l-1)(2l-3)}{2l(2l-2)(2l-4)} a^{s-4} + \ldots +$$

$$+(-1)^{l-1} \frac{(2l-1)(2l-3) \ldots 3}{2l(2l-2)(2l-4) \ldots 2} \omega^* a^{s-2l} +$$

$$+(-1)^{l} \frac{(2l-1)(2l-3) \ldots 3}{2l(2l-2)(2l-4) \ldots 2} l^{2l} I_0 .$$

(6)

It implies that the function $F$ in equation (5) is:

$$F(k') = \frac{2}{\pi} K(k') = \sum_{j=0}^{\infty} (b_j^-)^2 k^{2j} ,$$

(7)

once the first series representation (2) has been used. It is worth noting that $F(k') \to 1$ as $k' \to 0$: the last term in equation (5) is the one responsible for the logarithmic singularity of $K$ as $k \to 1^-$. The integral (6) implies also that the coefficients $s_j$ ($j = 0, 1, \ldots$) in equation (5) are linear combinations of the following series (see A):

$$\sigma^-_0 = \sum_{l=1}^{\infty} \frac{b_l^-}{l} = -2 \log \frac{\sqrt{2}+1}{2}$$

$$\sigma^-_j = \sum_{l=j+1}^{\infty} \frac{b_l^-}{l-j} = \sum_{j=0}^{j-1} b_j^- + b_j^- \left( \sum_{l=1}^{j} \frac{b_l^+ - \sqrt{2}}{l b_l^+} - 2 \log \frac{\sqrt{2}+1}{2} \right)$$

(8)

for $j \geq 1$. They are given by the formulæ:

$$s_0 = \frac{\sigma^-_0}{2} , \quad s_j = \frac{(-1)^j}{2} \left[ \sum_{m=0}^{j-1} a_m(j) \left( \sigma^-_m - \sum_{p=m+1}^{j} \frac{b_p^-}{p-m} \right) + a_j(j) \sigma^-_j \right]$$

for $j \geq 1$, where the numbers $a_m(j)$ ($j \geq 1$) are defined by introducing the quantities $\pi_p(l) := 1 + 1/[2(l-p)]$ in the following way:

$$a_0(j) = \prod_{p=1}^{j} \pi_p(0) , \quad a_m(j) = \frac{1}{2m} \prod_{p=1}^{j} \pi_p(m) \text{ for } m = 1, 2, \ldots, j .$$

The integral $K''$ in the decomposition (4) is evaluated by changing the integration variable from $t$ to $\eta$, with $t = (1 - k'' \eta^2)^{1/2}$, and using the
binomial series:

\[ K''(k) = \int_0^{t^*} \frac{d\eta}{(1 + k^2\eta^2)^{1/2}} \frac{1}{(1 - k^2\eta^2)^{1/2}} \]

(9)

\[ = \sum_{j=0}^{\infty} (-1)^j b_j^2 k^{2j} \int_0^{t^*} d\eta \frac{\eta^{2j}}{\sqrt{1 + k^2\eta^2}}, \]

where the integrals are evaluated by means of the recurrence relations: 

\[ M_0(k) = \log([\sqrt{2} + k]/(\sqrt{2} - k))/2 \]

and 

\[ M_j(k) = t^*2j/(\sqrt{2}jk^2) - (j - 1/2)M_{j-1}(k)/(jk^2) \]

(j \geq 1), or explicitly:

\[ M_j(k) = \frac{b_j^2}{k^{2j}} \left[ M_0(k) + \frac{1}{\sqrt{2}} \sum_{m=1}^j \frac{k^{2(m-1)t^*2m}}{m b_m} \right]. \]

The numerical approximation of the asymptotic series obtained by joining the expansions (5, 7, 9) will be shortly discussed in Section 4. It will be also compared with the one in Literature (see equation 1b in [4]).

3. Complete elliptic integral of second kind.

An analogous procedure is used in deducing the representation of the complete elliptic integral of second kind. The integral is firstly divided in the sum of two integrals:

\[ E(k) = \int_0^{t^*} dt \frac{1 - t^2 + k^2t^2}{(1 - t^2)^{1/2}} + \int_{t^*}^1 dt \frac{1 - k^2t^2}{(1 - t^2)^{1/2}}, \]

(10)

where \(1 - t^2 > k^2t^2\) holds in the first one (\(E'\)), while in the second integral (\(E''\)) one has \(k^2t^2 > 1 - t^2\).

The integrand function inside \(E'\) is expanded in binomial series in the ratio \(1 + k^2t^2/(1 - t^2)\):

\[ E'(k) = \sum_{l=0}^{\infty} b_l^2 \int_0^{t^*} dt \frac{t^{2l}}{(1 - t^2)^l} = t^* \left( 1 + \sum_{j=1}^{\infty} \sum_{l=0}^{\infty} v_j k^{2j} \right) + q(k')G(k'), \]

(11)

the integrals involved being evaluated by means of the recurrence relations 

\[ J_0(k') = t^*, \ J_1(k') = k'^2 q(k'), \ J_l(k') = [k'^2t^* - (2l - 1)k'^2J_{l-1}(k')]/[2(l - 1)] \]
\((l \geq 2)\), with \(q(k') := -\log k' + \log(1 + \omega^*) - t^*\). They lead to the integrals:

\[
J_2(k') = \frac{1}{2} k'^2 t^* - \frac{3}{2} k'^4 q(k')
\]

\[
J_3(k') = \frac{1}{4} k'^2 t^* - \frac{5}{8} k'^4 t^* + \frac{15}{8} k'^6 q(k')
\]

and so on. An inspection to the above sequence enables to write the \(l\)th \((l \geq 2)\) integral by the following formula:

\[
J_l(k') = \frac{1}{2(l-1)} t^* k'^2 - \frac{2l-1}{2(l-1)2(l-2)} t^* k'^4 + \frac{2l-1}{2(l-1)2(l-2)2(l-3)} t^* k'^6 + \ldots + \\
+ (-1)^{l-1} \frac{(2l-1)(2l-3) \ldots \cdot 5}{2(l-1)2(l-2)2(l-3) \ldots \cdot 2} t^* k'^{2(l-1)} + \\
+ (-1)^{l+1} \frac{(2l-1)(2l-3) \ldots \cdot 5 \cdot 3}{2(l-1)2(l-2)2(l-3) \ldots \cdot 2} k'^2 q(k').
\] (12)

The above general form of the integral \(J_l\) gives the function \(G\) in equation (11):

\[
G(k') = \frac{2}{\pi} [K(k') - E(k')] = \sum_{j=1}^{\infty} \frac{j}{j-1/2} b_j^2.k'^j.
\] (13)

Moreover, the integral (12) implies that the coefficients \(v_j\) \((j = 1, 2, \ldots)\) in the same equation are linear combinations of the following series (see A):

\[
\sigma_j^+ := \sum_{l=j+1}^{\infty} b_l^+ = \sum_{l=j}^{j-1} b_l^+ + b_j^+ \left[ \sum_{l=1}^{j} \frac{c_l - 2^{j/2}}{l b_l^+} + 2 \left( \sqrt{2} - 1 - \log \sqrt{2} + \frac{1}{2} \right) \right].
\] (14)

They are given by the formulae:

\[
v_1 = \frac{\sigma_1^+}{2}
\]

\[
v_j = \frac{(-1)^{j+1}}{2} \left[ \sum_{m=1}^{j-1} d_m(j) \left( \sigma_m^+ - \sum_{p=m+1}^{j} \frac{b_p^+}{p-m} \right) + d_j(j) \sigma_j^+ \right] \text{ for } j \geq 2,
\]

where the numbers \(d_m(j)\) \((j \geq 2)\) are calculated by introducing the quantities \(\chi_p(l) := 1 + 3/[2(l - p)]\) as follows:

\[
d_1(j) = \prod_{p=2}^{j} \chi_p(1), \quad d_m(j) = \frac{3}{2(m-1)} \prod_{p=2}^{j} \chi_p(m) \text{ for } m = 2, 3, \ldots, j.
\]
As in Section 2, the integration variable in the integral $E''(10)$ is changed from $t$ to $\eta$, with $t = (1 - k'^2 \eta^2)^{1/2}$, and the binomial series is used, leading to the representation of $E''$:

\[
E''(k) = k'^2 \int_0^\infty d\eta \left( \frac{1 + k^2 \eta^2}{1 - k'^2 \eta^2} \right)^{1/2} \int_0^{\eta^*} d\eta' \eta'^2 (1 + k^2 \eta'^2)^{-1/2},
\]

where the integrals $N_j(k)$ follow from the recurrence relations: $N_0(k) = \{2\sqrt{2}k/(2 - k^2) + \log\left(\sqrt{2} + k/(\sqrt{2} - k)\right)/4k\}$ and $N_j(k) = \sqrt{2}k^{2j+1} - (j-1/2)N_{j-1}(k)/[(j+1)k^2]$ ($j \geq 1$). As a consequence, they are also given by the explicit formulae:

\[
N_j(k) = \frac{b_j^{+1}}{k^{2j}} \left[ 2N_0(k) + \sqrt{2} \sum_{m=1}^j \frac{k^{2(m-1)}k^{2(m+1)}}{(m+1) b_{m+1}^+} \right].
\]

The asymptotic series obtained by joining the expansions (11, 13, 15) will be numerically compared with the one in Literature (see equation $\Pi_b$ in [4]) in the next section.

4. Numerical tests and some concluding remarks.

In the present section the asymptotic expansions in the complementary modulus $k'$ for the first ($K$) and the second ($E$) complete elliptic integrals obtained in Sections 2 and 3 are compared with the ones in Literature (e.g., see [4] equations $I_b$ and $\Pi_b$).

Once the order $j_{\text{max}}$ of the asymptotic expansions has been fixed, the present ones ($K_p^{(a)}$, $E_p^{(a)}$) and the corresponding asymptotic expansions ($K_L^{(a)}$, $E_L^{(a)}$) in [4] are computed on a narrow left neighbourhood of the point $k = 1$. The differences $|K_p^{(a)} - K_L^{(a)}|$ and $|E_p^{(a)} - E_L^{(a)}|$ are named $e_{pL}$ and drawn in Fig. 2 and 3 with green lines. They are about two orders of magnitude smaller than the errors due to the truncation of the asymptotic series ($\epsilon_i := k'^2(j_{\text{max}}+1)$, drawn with black lines). For a small $j_{\text{max}}$ ($j_{\text{max}} = 5$ in Fig. 2a, 3a) they lie below the machine precision (about $10^{-16}$) just before the point $k = 1$. As expected, at higher orders ($j_{\text{max}} = 10$ in Fig. 2b, 3b) these differences are smaller and there are large left neighbourhoods of the point $k = 1$ in which they lie below the machine precision. Moreover, the present asymptotic expansions are easy to program as the ones in [4] and have roughly the same computational cost. For this reason one can
obviously conclude that they behave as the existing ones, from a computational point of view. As already stressed in Section 1, their importance lies in the elementary way in which they have been deduced.

The use of the present asymptotic expansions \((K_p^{(a)}, E_p^{(a)})\), as well as of the ones in Literature \((K_L^{(a)}, E_L^{(a)})\), is also compared with the direct computation of the series in \(k\) \((2)\). As discussed in Section 1, they have been approximated by truncating terms smaller than \(10^{-16}\); the corresponding partial sums are named \(K\) and \(E\) and are very large, as shown in Fig. 1. The differences \(|K - K_p^{(a)}|\), \(|E - E_p^{(a)}|\) and \(|K - K_L^{(a)}|\), \(|E - E_L^{(a)}|\), called \(e_p\) and \(e_L\), are drawn in Fig. 2, 3 with red and blue lines, respectively. These curves explain better than any discussion the difficulties in summing the series \((2)\) near \(k = 1\). Indeed, note that they are close in their smooth parts, while the rough oscillations near \(k = 1\) are almost superimposed. It means that the oscillations affect the numerical approximations of the series \((2)\) and not the ones \((K_p,L^{(a)}, E_p,L^{(a)})\) of the asymptotic series. As a consequence, the use of the series \((2)\) in a small left neighbourhood of \(k = 1\) leads also to a loss of accuracy, in addition to the one of computational efficiency already shown in Fig. 1.
A. On the sum of the series (8, 14).

The sum of the series (8) is found in terms of the following integral representation:

\begin{equation}
\sigma_j^- = \int_0^1 \frac{dx}{x^{j+1}} \left( \frac{1}{\sqrt{1+x}} - \sum_{m=0}^j b_m^- x^m \right).
\end{equation}

It is handled by evaluating the indefinite integrals (additive constants are omitted):

\[ y_j := \int \frac{dx}{x^{j+1}} \frac{1}{\sqrt{1+x}} \]

by means of the recurrence formulae \((j \geq 1)\):

\[ y_0 = \log \frac{\sqrt{x} + \sqrt{1+x} - 1}{\sqrt{x} + \sqrt{1+x} + 1} \]

\[ y_j = -\frac{1}{j} \frac{\sqrt{1+x}}{x^j} - \frac{j-1/2}{j} y_{j-1} = b_j^- \left( y_0 - \sqrt{1+x} \sum_{l=1}^j \frac{1}{l b_l^-} \frac{1}{x^l} \right). \]

The above integrals are inserted into the representation (16) and, by accounting for the identities:

\begin{equation}
\sum_{l=m}^j \frac{b_l^{j-m}}{l b_l^j} = \frac{b_j^{j-m}}{m b_j^j} \quad \text{for} \quad m = 1, 2, \ldots, j,
\end{equation}

Figure 3. As in Fig. 2, but for \(E\).
the coefficients of the non-integrable terms $1/x^m$ vanish so that the sum (8) follows.

The identities (17) are proved by induction. They are verified for $j = 1$. Assumed that they are verified for a certain $j (> 1)$, it is now shown that they hold also for $j + 1$. Indeed, they are easily verified for $m = j + 1$, while for $m = 1, 2, \ldots, j$ the above sum is rewritten as:

$$\sum_{l=m}^{j+1} b_{l-m}^+ b_l^- = \frac{b_{j-m}^-}{m b_j^- (j+1) b_{j+1}^-} + \frac{b_{j+1-m}^+}{(j+1) b_{j+1}^-}$$

$$= \frac{(-1)^m 2^{2m} (j!)^2 [2(j-m)]!}{m [(j-m)!]^2 (2j)!} + \frac{(-1)^{m-1} 2^{2m+1} j! (j+1)! [2(j-m)]!}{(j-m)! (j+1-m)! [2(j+1)]!}$$

$$= \frac{(-1)^m 2^{2m} [(j+1)!]^2 [2(j+1-m)]!}{m [(j+1-m)!]^2 [2(j+1)]!} = \frac{b_{j+1-m}^-}{m b_{j+1}^-},$$

according to the formulae (17) with $j + 1$ replacing $j$.

The sum (14) is evaluated in an analogous way. It is firstly rewritten in the integral form:

$$(18) \quad \sigma_j^+ = \int_0^1 \frac{dx}{x^{j+1}} \left( \sqrt{1+x} - \sum_{m=0}^j b_m^+ x^m \right).$$

Then the indefinite integrals:

$$z_j := \int dx \sqrt{1+x}$$

are evaluated by means of the recurrence formulae ($j \geq 1$):

$$z_0 = 2 \left( \sqrt{1+x} + \log \frac{\sqrt x + \sqrt{1+x} - 1}{\sqrt x + \sqrt{1+x} + 1} \right)$$

$$z_j = -\frac{1}{j} \frac{(1+x)^{3/2}}{x^j} - \frac{j-3/2}{j} z_{j-1} = b_j^+ \left[ z_0 - (1+x)^{3/2} \sum_{l=1}^j \frac{1}{l b_l^-} \frac{1}{x^l} \right].$$

The above integrals are inserted into the representation (18) and the identities:

$$(19) \quad \sum_{l=m}^{j} c_{l-m}^- \frac{b_{l-m}^+}{m b_j^+} = \frac{b_{j-m}^-}{m b_j^-} \quad \text{for} \quad m = 1, 2, \ldots, j,$$
are used in order to cancel the non-integrable terms $1/x^m$. In this way the sum (14) follows.

The identities (19) are proved by induction, as before. They are verified for $j = 1$. Assumed that they hold for a certain $j (> 1)$, it is proved that they are verified for $j + 1$. Indeed, they hold for $m = j + 1$, while for $m = 1, 2, \ldots, j$ the above sum becomes:

$$
\sum_{l=m}^{j+1} \frac{c_{l-m}}{l} b_l^+ = \frac{b_{j-m}^+}{m b_j^+} + \frac{c_{j+1-m}}{(j+1) b_{j+1}^+} = \frac{(-1)^m 2^m (2j+1) (j)!^2 [2(j+1-m)]!}{m (2j - 2m - 1) [(j-m)]!^2 (2j)!} + \frac{(-1)^{m+1} 2^{2m} 3 (j!)^2 [2(j-m)]!}{(2j - 2m - 1) [(j-m-1)]!^2 (2j)!} + \frac{(-1)^m 2^{2m} (2j+1) [(j+1)]!^2 [2(j+1-m)]!}{m (2j - 2m + 1) [(j+1-m)]!^2 (2j+1)!} = \frac{b_{j+1-m}^+}{m b_{j+1}^+},
$$

according to the identities (19) with $j + 1$ replacing $j$.

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