Fractional heat conduction in a semi-infinite composite body

Yuriy Povstenko\textsuperscript{1,2}

\textsuperscript{1}Institute of Mathematics and Computer Science, Jan Długosz University in Częstochowa, Poland
\textsuperscript{j.povstenko@ajd.czest.pl}

\textsuperscript{2}Department of Informatics, European University of Information Technology and Economics, Warszaw, Poland

Dedicated to Professor Francesco Mainardi on the occasion of his retirement

Communicated by Gianni Pagnini and Enrico Scalas

Abstract

A medium consisting of a region $0 < x < L$ and a region $L < x < \infty$ is considered. Heat conduction in one region is described by the equation with the Caputo time-fractional derivative of order $\alpha$, whereas heat conduction in another region is described by the equation with the time derivative of the order $\beta$. The problem is solved under conditions of perfect contact, i.e. when the temperatures at the contact point and the heat fluxes through the contact point are the same for both regions. The solution valid for small values of time is expressed in terms of the Mittag-Leffler function and the Mainardi function. Several particular cases are considered and illustrated graphically.

Keywords: non-Fourier heat conduction, fractional calculus, Mainardi function, Mittag-Leffler function.

AMS subject classification: 26A33, 35K05, 45K05.

1. Introduction.

The classical theory of heat conduction is based on the Fourier law

\begin{equation}
q = -k \text{grad} T,
\end{equation}

where $q$ is the heat flux vector, $T$ denotes temperature, and $k$ is the thermal conductivity. In combination with the law of conservation of energy, the standard Fourier law results in the parabolic heat conduction equation.
The time-nonlocal dependence between the heat flux vector and the temperature gradient with the “long-tail” power kernel \([1]–[3]\) can be interpreted in terms of fractional calculus

\[
q(t) = -kD^{1-\alpha}_{RL} \text{grad} T(t), \quad 0 < \alpha \leq 1,
\]

\[
q(t) = -kI^{\alpha-1} \text{grad} T(t), \quad 1 < \alpha \leq 2.
\]

Here \(I^\alpha f(t)\) and \(D^\alpha_{RL} f(t)\) are the Riemann–Liouville fractional integral and derivative of the order \(\alpha\), respectively, \([4]–[7]\):

\[
I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \, d\tau, \quad \alpha > 0,
\]

\[
D^\alpha_{RL} f(t) = \frac{d^m}{dt^m} \left[ \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau) \, d\tau \right],
\]

where \(m - 1 < \alpha < m\).

It should be noted that in fractional calculus, where integrals and derivatives of arbitrary (not integer) order are considered, there is no sharp boundary between integration and differentiation. For this reason, some authors \([6], [8]\) do not use a separate notation for the fractional integral \(I^\alpha f(t)\). The fractional integral integral \(I^\alpha f(t)\) of the order \(\alpha > 0\) is denoted as \(D^\alpha_{RL} f(t)\). Using this notation, Equations (2) and (3) can be rewritten as one dependence

\[
q(t) = -kD^{1-\alpha}_{RL} \text{grad} T(t), \quad 0 < \alpha \leq 2.
\]

In combination with the law of conservation of energy, the constitutive equation (6) leads to the time fractional heat conduction equation

\[
\frac{\partial^\alpha T}{\partial t^\alpha} = a \Delta T, \quad 0 < \alpha \leq 2,
\]

with the Caputo fractional derivative

\[
\frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m f(\tau)}{dT^m} \, d\tau, \quad m - 1 < \alpha < m.
\]

Starting from the pioneering papers \([9]–[13]\), considerable interest has been shown in solutions to Equation (7). Different kinds of boundary conditions for time-fractional heat conduction equation were analyzed in \([14], [15]\).
If the surfaces of two solids are in perfect thermal contact, the temperatures on the contact surface and the heat fluxes through the contact surface are the same for both solids, and we obtain the boundary conditions of the fourth kind:

\[ T_1|_S = T_2|_S, \]

\[ k_1 D_{RL}^{1-\alpha} \frac{\partial T_1}{\partial n}|_S = k_2 D_{RL}^{1-\beta} \frac{\partial T_2}{\partial n}|_S, \quad 0 < \alpha \leq 2, \quad 0 < \beta \leq 2, \]

where subscripts 1 and 2 refer to solids 1 and 2, respectively, and \( n \) is the common normal at the contact surface.

Classical heat conduction in composite medium was considered by many authors (see, for example, [16]–[20]). In the previous papers [15], [21], the problem of fractional heat conduction in two semi-infinite regions, \( x > 0 \) and \( x < 0 \), was investigated. In [22], the central-symmetric problem was studied for a composite medium consisting of a spherical inclusion \( 0 < r < R \) and a matrix \( R < r < \infty \). In the present paper, we consider a composite medium consisting of a region \( 0 < x < L \) and a region \( L < x < \infty \). Heat conduction in one region is described by the equation with the Caputo time-fractional derivative of order \( \alpha \), whereas heat conduction in another region is described by the equation with the time derivative of the order \( \beta \).

The problem for uniform initial temperature in a layer \( 0 < x < L \) and zero initial temperature in a region \( L < x < \infty \) is solved under the conditions of perfect contact at \( x = L \) and the insulation condition at the boundary surface \( x = 0 \). The solution is expressed in terms of the Mittag-Leffler function and the Mainardi function.


Recall the Laplace transform rules for fractional integrals and derivatives [5]–[7]:

\[ \mathcal{L}\{I^\alpha f(t)\} = \frac{1}{s^\alpha} f^*(s), \]

\[ \mathcal{L}\{D_{RL}^\alpha f(t)\} = s^\alpha f^*(s) \]

\[ - \sum_{k=0}^{m-1} D^k I^{m-k-\alpha} f(0^+) s^{m-1-k}, \quad m - 1 < \alpha < m, \]
\[ \mathcal{L} \left\{ \frac{d^\alpha f(t)}{dt^\alpha} \right\} = s^\alpha f^*(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-1-k}, \quad m - 1 < \alpha < m, \]

where \( s \) is the Laplace transform variable, the asterisk denotes the transform.

The Mittag-Leffler function in one parameter \( \alpha \) [5]–[7]
\[ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \quad z \in \mathbb{C}, \]
provides a generalization of the exponential function.

The Mittag-Leffler type function in two parameters \( \alpha \) and \( \beta \) [5]–[7] is described by the following series representation:
\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta > 0, \quad z \in \mathbb{C}. \]

The essential role of the Mittag-Leffler functions in fractional calculus results from the formula for the inverse Laplace transform [6]
\[ \mathcal{L}^{-1} \left\{ \frac{s^{\alpha-\beta}}{s^{\alpha} + b} \right\} = t^{\beta-1} E_{\alpha,\beta}(-bt^\alpha). \]

The Wright function is defined as [6], [7], [12], [13]
\[ W(\alpha, \beta; z) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(\alpha k + \beta)}, \quad \alpha > -1, \quad z \in \mathbb{C}. \]

The Mainardi function \( M(\alpha; z) \) [6], [12], [13] is the particular case of the Wright function
\[ M(\alpha; z) = W(-\alpha, 1 - \alpha; -z) \]
\[ = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k! \Gamma[-\alpha k + (1 - \alpha)]}, \quad 0 < \alpha < 1, \quad z \in \mathbb{C}. \]

The Mainardi and Wright functions appear in the formulae for the inverse Laplace transform (see [12], [13], [23]–[26])
\[ \mathcal{L}^{-1} \{ \exp (-\lambda s^\alpha) \} = \frac{\alpha \lambda}{\lambda^\alpha + 1} M(\alpha; \lambda t^{-\alpha}), \quad 0 < \alpha < 1, \quad \lambda > 0, \]
\[ \mathcal{L}^{-1}\left\{ s^{-\alpha} \exp(-\lambda s^\alpha)\right\} = t^{-\alpha} M(\alpha; \lambda t^{-\alpha}), \quad 0 < \alpha < 1, \quad \lambda > 0, \]

\[ \mathcal{L}^{-1}\left\{ s^{-\beta} \exp(-\lambda s^\alpha)\right\} \]
\[ = t^{\beta-1} W(-\alpha, \beta; -\lambda t^{-\alpha}), \quad 0 < \alpha < 1, \quad \lambda > 0. \]


Consider the time-fractional heat conduction equations with the Caputo derivative in a two-layer medium composed of a region \( 0 < x < L \) and a region \( L < x < \infty \):

\[ \frac{\partial^\alpha T_1}{\partial t^\alpha} = a_1 \frac{\partial^2 T_1}{\partial x^2} + \Phi_1(x, t), \quad 0 < x < L, \quad 0 < \alpha \leq 2, \]

\[ \frac{\partial^\beta T_2}{\partial t^\beta} = a_2 \frac{\partial^2 T_2}{\partial x^2} + \Phi_2(x, t), \quad L < x < \infty, \quad 0 < \beta \leq 2, \]

under the initial conditions

\[ t = 0 : \quad T_1 = f_1(x), \quad 0 < x < L, \quad 0 < \alpha \leq 2, \]

\[ t = 0 : \quad \frac{\partial T_1}{\partial t} = F_1(x), \quad 0 < x < L, \quad 1 < \alpha \leq 2, \]

\[ t = 0 : \quad T_2 = f_2(x), \quad 0 < x < \infty, \quad 0 < \beta \leq 2, \]

\[ t = 0 : \quad \frac{\partial T_2}{\partial t} = F_2(x), \quad 0 < x < \infty, \quad 1 < \beta \leq 2, \]

and the boundary conditions of perfect thermal contact

\[ x = L : \quad T_1(x, t) = T_2(x, t), \]

\[ x = L : \quad k_1 D_{RL}^{1-\alpha} \frac{\partial T_1(x, t)}{\partial x} = k_1 D_{RL}^{1-\beta} \frac{\partial T_2(x, t)}{\partial x}, \quad 0 < \alpha \leq 2, \quad 0 < \beta \leq 2. \]

The boundary surface \( x = 0 \) is kept insulated:

\[ x = 0 : \quad \frac{\partial T(x, t)}{\partial x} = 0. \]

In addition, the boundedness condition at infinity is assumed

\[ \lim_{x \to \infty} T(x, t) = 0. \]
4. Uniform initial temperature in the layer.

In this case we consider the time-fractional heat conduction equations

\[ \frac{\partial^\alpha T_1}{\partial t^\alpha} = a_1 \frac{\partial^2 T_1}{\partial x^2}, \quad 0 < x < L, \quad 0 < \alpha \leq 2, \]

\[ \frac{\partial^\beta T_2}{\partial t^\beta} = a_2 \frac{\partial^2 T_2}{\partial x^2}, \quad L < x < \infty, \quad 0 < \beta \leq 2, \]

under the initial conditions

\[ t = 0 : \quad T_1 = T_0, \quad 0 < x < L, \quad 0 < \alpha \leq 2, \]

\[ t = 0 : \quad \frac{\partial T_1}{\partial t} = 0, \quad 0 < x < L, \quad 1 < \alpha \leq 2, \]

\[ t = 0 : \quad T_2 = 0, \quad 0 < x < \infty, \quad 0 < \beta \leq 2, \]

\[ t = 0 : \quad \frac{\partial T_2}{\partial t} = 0, \quad 0 < x < \infty, \quad 1 < \beta \leq 2, \]

and the boundary conditions (28)–(30).

The Laplace transform with respect to time gives two ordinary differential equations

\[ s^\alpha T_1^s - s^{\alpha - 1} T_0 = a_1 \frac{d^2 T_1^s}{dx^2}, \quad 0 < x < L, \]

\[ s^\beta T_2^s = a_2 \frac{d^2 T_2^s}{dx^2}, \quad L < x < \infty, \]

and the boundary conditions

\[ x = 0 : \quad \frac{dT_1^s}{dx} = 0, \]

\[ x = L : \quad T_1^s = T_2^s, \]

\[ x = L : \quad k_1 s^{1-\alpha} \frac{dT_1^s}{dx} = k_2 s^{1-\beta} \frac{dT_2^s}{dx}, \]

\[ x \to \infty : \quad T_2^s = 0. \]

The solutions of equations (38) and (39) have the following form:

\[ T_1^s = \frac{T_0}{s} + A \sinh \left( \sqrt{\frac{s^\alpha}{a_1}} x \right) + B \cosh \left( \sqrt{\frac{s^{\alpha/2}}{a_1}} x \right), \quad 0 < x < L, \]
From conditions (40) and (43) it follows that

\[ T_2^* = C \exp \left( -\sqrt{\frac{s^3}{a_2}}x \right) + D \exp \left( \sqrt{\frac{s^3}{a_2}}x \right), \quad L < x < \infty. \]

(46) \quad B = 0, \quad D = 0,

whereas the conditions of the perfect thermal contact (42) and (43) give

\[ A = -\frac{T_0}{s} \frac{1}{\cosh \left( \sqrt{\frac{s^3}{a_1}}L \right) + \gamma s^{\beta/2-\alpha/2} \sinh \left( \sqrt{\frac{s^3}{a_1}}L \right)}, \]

\[ C = \frac{T_0}{s} \exp \left( \sqrt{\frac{s^3}{a_2}}L \right) \]

(47) \quad - \frac{T_0}{s} \frac{\cosh \left( \sqrt{\frac{s^3}{a_1}}L \right) \exp \left( \sqrt{\frac{s^3}{a_2}}L \right)}{\cosh \left( \sqrt{\frac{s^3}{a_1}}L \right) + \gamma s^{\beta/2-\alpha/2} \sinh \left( \sqrt{\frac{s^3}{a_1}}L \right)}, \]

where

\[ \gamma = \frac{k_1 \sqrt{a_2}}{k_2 \sqrt{a_1}}. \]

Hence,

\[ T_1^* = \frac{T_0}{s} - \frac{T_0}{s} \frac{\cosh \left( \sqrt{\frac{s^3}{a_1}}x \right)}{\cosh \left( \sqrt{\frac{s^3}{a_1}}L \right) + \gamma s^{\beta/2-\alpha/2} \sinh \left( \sqrt{\frac{s^3}{a_1}}L \right)}, \]

\[ T_2^* = \frac{T_0}{s} \exp \left[ -\sqrt{\frac{s^3}{a_2}}(x - L) \right] \]

(48) \quad - \frac{T_0}{s} \frac{\cosh \left( \sqrt{\frac{s^3}{a_1}}L \right) \exp \left[ -\sqrt{\frac{s^3}{a_2}}(x - L) \right]}{\cosh \left( \sqrt{\frac{s^3}{a_1}}L \right) + \gamma s^{\beta/2-\alpha/2} \sinh \left( \sqrt{\frac{s^3}{a_1}}L \right)}. \]

Now we will investigate the approximate solution of the considered problem for small values of time. In the case of classical heat conduction equation
Y. Povstenko

this method was described in [16], [18]. Based on the Tauberian theorems for the Laplace transform, for small values of time $t$ (the large values of the transform variable $s$) we can neglect the exponential term in comparison with 1:

$$1 \pm \exp \left( -2 \sqrt{\frac{s}{a_1}} L \right) \simeq 1,$$

thus obtaining

$$T_1^* \simeq \frac{T_0}{s} \left\{ 1 - \frac{\exp \left[ -\sqrt{\frac{s}{a_1}} (L - x) \right]}{1 + \gamma s^{\beta/2 - \alpha/2}} \right\},$$

$$T_2^* \simeq \frac{T_0}{s} \frac{\gamma s^{\beta/2 - \alpha/2}}{1 + \gamma s^{\beta/2 - \alpha/2}} \exp \left[ -\sqrt{\frac{s}{a_2}} (x - L) \right].$$

Inverting the Laplace transform, we get:

a) $\beta > \alpha$

$$T_1(x, t) \simeq T_0 - \frac{T_0}{\gamma} \int_0^t \frac{(t - \tau)^{\beta/2 - 1}}{\tau^{\alpha/2}} M \left( \frac{\alpha}{2}; \frac{L - x}{\sqrt{a_1} \tau^{\alpha/2}} \right)$$

$$\times E_{\beta/2 - \alpha/2, \beta/2} \left[ -\frac{1}{\gamma} (t - \tau)^{\beta/2 - \alpha/2} \right] d\tau, \quad 0 \leq x \leq L,$$

$$T_2(x, t) \simeq T_0 \int_0^t \frac{(t - \tau)^{\alpha/2 - 1}}{\tau^{\beta/2}} M \left( \frac{\beta}{2}; \frac{x - L}{\sqrt{a_2} \tau^{\beta/2}} \right)$$

$$\times E_{\beta/2 - \alpha/2, \alpha/2} \left[ -\frac{1}{\gamma} (t - \tau)^{\beta/2 - \alpha/2} \right] d\tau, \quad L \leq x < \infty.$$  

b) $\alpha > \beta$

$$T_1(x, t) \simeq T_0 - T_0 \int_0^t \frac{(t - \tau)^{\alpha/2 - 1}}{\tau^{\beta/2}} M \left( \frac{\alpha}{2}; \frac{L - x}{\sqrt{a_1} \tau^{\alpha/2}} \right)$$

$$\times E_{\alpha/2 - \beta/2, \alpha/2} \left[ -\gamma (t - \tau)^{\alpha/2 - \beta/2} \right] d\tau, \quad 0 \leq x \leq L,$$

$$T_2(x, t) \simeq T_0 \int_0^t \frac{(t - \tau)^{\alpha/2 - 1}}{\tau^{\beta/2}} M \left( \frac{\alpha}{2}; \frac{L - x}{\sqrt{a_1} \tau^{\alpha/2}} \right)$$

$$\times E_{\alpha/2 - \beta/2, \alpha/2} \left[ -\gamma (t - \tau)^{\alpha/2 - \beta/2} \right] d\tau, \quad L \leq x < \infty.$$
Figure 1. Dependence of the solution in a semi-infinite composite body on distance.

\[ T_2(x,t) \simeq T_0 \gamma \int_0^t \frac{(t-\tau)^{\alpha/2-1}}{\tau^{\beta/2}} M \left( \frac{\beta}{2}, \frac{x-L}{\sqrt{a_1} \tau^{\beta/2}} \right) \times E_{\alpha/2-\beta/2, \alpha/2} \left[ -\gamma (t-\tau)^{\alpha/2-\beta/2} \right] \, d\tau, \quad L \leq x < \infty. \]  

For example, for \( \alpha = 2, \beta = 1 \)

\[ T_1 \simeq \begin{cases}  
T_0, & 0 \leq x < L - \sqrt{a_1} t, \\
T_0 \left\{ 1 - \exp \left[ \gamma^2 \left( t - \frac{L-x}{\sqrt{a_1}} \right) \right] \erfc \left( \gamma \sqrt{t - \frac{L-x}{\sqrt{a_1}}} \right) \right\}, & L - \sqrt{a_1} t < x < L; 
\end{cases} \]

\[ T_2 \simeq T_0 \left[ \erfc \left( \frac{x-L}{2\sqrt{a_2} t} \right) - \exp \left( \gamma \frac{x-L}{\sqrt{a_2}} + \gamma^2 t \right) \right. \\ \times \left. \erfc \left( \frac{x-L}{2\sqrt{a_2} t} + \gamma \sqrt{t} \right) \right], \quad L < x < \infty. \]
Figure 2. Dependence of the solution in a semi-infinite composite body on distance.

For $\alpha = 1$, $\beta = 2$

\[
T_1 \simeq T_0 \left[ \text{erf} \left( \frac{L - x}{2\sqrt{a_1 t}} \right) + \exp \left( \frac{L - x}{\gamma \sqrt{a_1}} + \frac{t}{\gamma^2} \right) \right] \\
\times \text{erfc} \left( \frac{L - x}{2\sqrt{a_1 t}} + \frac{\sqrt{t}}{\gamma} \right), \quad 0 \leq x < L,
\]

(60)

\[
T_2 \simeq \begin{cases} 
T_0 \exp \left[ \frac{1}{\gamma^2} \left( t - \frac{x - L}{\sqrt{a_2}} \right) \right] \text{erfc} \left( \frac{1}{\gamma} \sqrt{\frac{t - x - L}{\sqrt{a_2}}} \right), & L < x < L + \sqrt{a_2}t, \\
0, & L + \sqrt{a_1}t < x < \infty.
\end{cases}
\]

(61)

In particular, if $\alpha = \beta$, then

\[
T_1 \simeq T_0 - \frac{T_0}{1+\gamma} W \left( -\frac{\alpha}{2}, 1; -\frac{L - x}{\sqrt{a_1 t}^{\alpha/2}} \right), \quad 0 \leq x \leq L,
\]

(62)

\[
T_2 \simeq \frac{T_0 \gamma}{1+\gamma} W \left( -\frac{\alpha}{2}, 1; -\frac{x - L}{\sqrt{a_2 t}^{\gamma/2}} \right), \quad L \leq x < \infty.
\]

(63)
Several results of numerical calculations are presented in Figures 1–3 with the following nondimensional quantities:

\[ \frac{T}{T_0}, \quad \bar{x} = \frac{x}{L}, \quad \kappa = \frac{\sqrt{a_1 t^{\alpha/2}}}{L}, \quad \bar{\gamma} = \gamma t^{\alpha/2-\beta/2}, \quad \epsilon = \frac{\sqrt{a_1 \sqrt{a_2} t^{\alpha/2-\beta/2}}}{L}. \]

In calculations we have taken \( \kappa = 0.2, \bar{\gamma} = 2, \) and \( \epsilon = 0.6. \) Such values of nondimensional parameters show the typical features of the solution.

5. Conclusions.

We investigated the solution to the time-fractional heat conduction equations with different orders of the Caputo time derivatives in a composite medium consisting of two regions being in perfect thermal contact. In the case \( 0 < \alpha < 1, \) the time-fractional heat conduction equation interpolates the elliptic Helmholtz equation (\( \alpha \to 0 \)) and the parabolic heat conduction equation (\( \alpha = 1 \)). When \( 1 < \alpha < 2, \) the time-fractional heat conduction equation interpolates the standard heat conduction equation (\( \alpha = 1 \)) and the hyperbolic wave equation (\( \alpha = 2 \)). The Laplace transform with respect to time \( t \) reduces the problem to two ordinary differential equations which solutions have been analyzed for large values of the Laplace transform variable. Based on the Tauberian theorems for Laplace transform, the approximate solution valid for small values of time was ob-
The numerical results for several values of the order of fractional derivatives show the dependence of the solution on distance.

REFERENCES
14. Y. Povstenko, Different kinds of boundary condition for time-fractional heat conduction equation, in *13th International Carpathian Con-


